MYRZAKUL AKBOTA RATBAYKYZY

Investigations of some integrable multilayer spin systems and their connection with multicomponent nonlinear Schrödinger equations

ABSTRACT
of the dissertation for the degree of Doctor of Philosophy (PhD) on specialty 6D060100 — Mathematics

Relevance of the research topic. In recent decades, there has been a growing interest in investigating the theory of integrable nonlinear evolution equations (INEE). This is due to the fact that it is INEE that have applications in various fields of natural sciences. For example, in differential geometry they can reproduce integrable classes of curves and surfaces, while in physics they describe the nonlinear dynamics of wave processes. The works of Ablowitz M., Sigur H., Lamb J., Dodd R., Newell A., Novikov S.P., Zakharov V.E., Shabat A.B., Lakshmanan M., Ding K. and others are devoted to the creation of the INEE theory.

The works of Bordag L., Yanovsky A., Martina L, Anko S. and others are devoted to the issues of the connection of single-layer INEE with differential geometry. The relevance of the investigations of multilayer INEE is substantiated by the development of modern science in this direction.

The purpose of the work. Investigation of multicomponent nonlinear evolution equations of the Schrödinger type and finding equivalent multilayer spin systems based on the INEE theory. Development of a generalized method for establishing geometric and gauge equivalence between single-layer INEE for multilayer ones, as well as finding their exact solutions for two-layer cases.

Research methodology. In the dissertation work, the well-known methods of the theory of solitons are taken as the basis of the research methodology as methods for establishing geometric and gauge equivalence between single-layer INEE and the Darboux transformation method for finding their exact solutions. Since multicomponent INEE are studied in the dissertation work, the above methods for studying single-component INEE are extended for multicomponent cases.

The main provisions of the dissertation submitted for defense and the results of the investigations:

The following research results obtained in the framework of the dissertation work are presented for defense:

– a new method for establishing equivalence between integrable nonlinear equations based on the properties of Lie algebras;

– geometric and gauge equivalence between two-layer integrable equations;

– Γ – spin system's exact solution and its connection with the solution of the two-layer spin system;

– a generalization of the method for establishing geometric equivalence between two-layer integrable equations for *n*-layer cases.

Theoretical and practical value. The work is theoretical in nature and will contribute to the further development of the theory of multilayer integrable nonlinear partial differential equations.

The structure and scope of the dissertation. The dissertation consists of an introduction, four sections, a conclusion, a list of references and an appendix.

The main content of the work.

The first section proposes for the first time a method for establishing equivalence between one-layer integrable nonlinear equations based on the isomorphism properties of the Lie algebras *su(2)* and *so(3)*. The method is demonstrated on the non-linear Schrödinger equations

$$iq_t + q_{xx} + 2\bar{q}q^2 = 0,$$

and the Heisenberg ferromagnet equations (HFE)

$$\vec{S}_t = \vec{S} \wedge \vec{S}_{xx},$$

as well as on Schrödinger-type equations and generalized HFE. Here $q(x,t) -$ complex-valued function, $\vec{S}(x,t) -$ spin vector.

In the second section, a two-layer spin system is derived

$$\begin{aligned} iA_t + \frac{1}{2}[A, A_{xx}] + iu_1 A_x + v_1[\sigma_3, A] &= 0, \\ iB_t + \frac{1}{2}[B, B_{xx}] + iu_2 B_x + v_2[\sigma_3, B] &= 0, \end{aligned}$$

where $u_j, v_j (j = 1,2)$ – connections potentials, A and B – spin matrices, and its geometric connection is installed with the two-component Manakov system

$$\begin{aligned} iq_{1t} + q_{1xx} + 2(|q_1|^2 + |q_2|^2)q_1 &= 0, \\ iq_{2t} + q_{2xx} + 2(|q_1|^2 + |q_2|^2)q_2 &= 0. \end{aligned}$$

In the third section, we show that the equivalent analogue of the multilayer spin system

$$\begin{aligned} &iA_t^{(1)} + \frac{1}{2}\left[A^{(1)}, A_{xx}^{(1)}\right] + iu_1 A_x^{(1)} + v_1\left[\sigma_3, A^{(1)}\right] = 0, \\ &iA_t^{(2)} + \frac{1}{2}\left[A^{(2)}, A_{xx}^{(2)}\right] + iu_2 A_x^{(2)} + v_2\left[\sigma_3, A^{(2)}\right] = 0, \\ &\vdots \\ &iA_t^{(N)} + \frac{1}{2}\left[A^{(N)}, A_{xx}^{(N)}\right] + iu_N A_x^{(N)} + v_N\left[\sigma_3, A^{(N)}\right] = 0, \end{aligned}$$

is the vector nonlinear Schrödinger equation

$$\begin{aligned} iq_{1t} + q_{1xx} + vq_1 &= 0, \\ iq_{2t} + q_{2xx} + vq_2 &= 0, \\ &\vdots \\ iq_{Nt} + q_{Nxx} + vq_N &= 0, \end{aligned}$$

where $v = 2\left(\sum_{j=1}^{N} |q_j|^2\right)$.

In the fourth section, solutions are found for the Γ-spin system, the two-layer spin system, and the generalized HFE.

The internal unity of the dissertation work. The results of the investigations are presented according to the principle "from simple to complex".

First, some one-layer (one-component) nonlinear integrable equations from various fields of mathematical physics are investigated, such as: the nonlinear Schrödinger equation, the Heisenberg ferromagnet model, then the two-component Manakov system and two-layer spin systems are considered. Further, the results of the study are generalized for *N*-component vector nonlinear Schrödinger equation.

Approbation of work. The results of the study were reported at international conferences as:

- XX International Conference "Geometry, Integrability, Quantization" (June 2-7, 2018. Varna, Bulgaria);
- XIV International scientific conference of students, undergraduates and young scientists "LOMONOSOV - 2018" (Astana, 2018);
- Republican scientific and practical conference "Modern problems of mathematical and computer modeling in the context of the development of the digital industry of Kazakhstan" (Astana, 2018).

Publications. As part of the research on the topic of the dissertation work, 7 works were published:

1. Surfaces and curves induced by nonlinear Schrodinger-type equations and their spin systems // Symmetry. – 2021. – Vol.13. – P.1827-1-1827-18 (WoS quartile – Q2, Scopus percentile – 90%).
2. Integrable motion of two interacting curves, spin systems and the Manakov system // International Journal of Geometric Methods in Modern Physics. – 2017. – Vol.14, №07. – P.1750115 (WoS quartile – Q2, Scopus percentile – 47%).
3. Integrable geometric flows of interacting curves/surfaces, multilayer spin systems and the vector nonlinear Schrödinger equation // International Journal of Geometric Methods in Modern Physics. – 2017. - Vol. 14, №10. – P.1750136(WoS quartile – Q2, Scopus percentile – 47%).
4. Integrability of the two-layer spin system // Geometry, Integrability and Quantization. – 2019. – Vol.20. –P.208–214. (Scopus percentile – 24%).
5. Интегрируемость двух взаимодействующих кривых и геометрически-эквивалентный спиновый аналог уравнения Манакова // Вестник ЕНУ им. Л.Н.Гумилева. – 2016. –Вып. 113, №4. – С. 23-26.
6. Интегрируемость двухпараметрического уравнения M-LIII // XIV международная научная конференция студентов, магистрантов и молодых ученых «Ломоносов – 2018». Тезисы докладов XIV международной научной конференции . - Астана, 2018. - Часть 1. - С.32-33.
7. Об интегрируемости спиновой системы с самосогласованным потенциалом // Современные проблемы математического и компьютерного моделирования в условиях развития цифровой индустрии Казахстана Сборник материалов республиканской научно-практической конференции. - Астана, 2018. - С.101-106.

Citation of scientific results. The main scientific results obtained within the framework of this dissertation are cited in the works of scientists from far abroad:

1. The third work from the above list of published works is cited three times by foreign authors: ZhaowenYan, Bian Gao, Minru Chen, Jifeng Cui in journal

«Chaos, Solitons and Fractals», doi.org/10.1016/j.chaos.2018.11.011; Nana Jiang, Meina Zhang, Jiafeng Guo, Zhaowen Yan in journal «Chaos, Solitons and Fractals», doi.org/10.1016/j.chaos.2020.109644; Rong Han, Haichao Sun, Nana Jiang, Zhaowen Yan in journal «Journal of Nonlinear Mathematical Physics», doi.org/10.1007/s44198-021-00001-0.

2. The results of the first article from the above list of published works are cited by the authors Rachel Klauss, Aaron Phillips and José M. Vega-Guzmán in journal «Symmetry», doi.org/10.3390/sym14030465.

Connection of this work with other research works. The dissertation work was carried out in accordance with the plans of research works of the following projects of grant funding of the Ministry of Education and Science of the Republic of Kazakhstan:

1) on the priority "Intellectual potential of the country" on the topic: "Investigation of the generalized Landau-Lifshitz equation with self-consistent sources and its integrable reductions" for 2015-2017;

2) scientific research in the field of natural sciences on the topic "Investigation of the connection between surfaces/manifolds geometry and integrable nonlinear evolution equations" for 2020-2022.